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LETTER TO THE EDITOR

Kadanoff's variational renormalisation group method: The Ising model on the square and triangular lattices

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Abstract. The renormalisation group method of Kadanoff is applied to the spin-$\frac{1}{2}$ Ising model in two dimensions using a transformation applicable to both the square and triangular lattices. Two types of cluster are studied for the triangular lattice. The first cluster yields a new fixed point but the results for the critical temperature and critical exponents are in poor agreement with the known exact values. The second cluster maps directly onto the fixed point found by Kadanoff for the square lattice yielding excellent values for both the critical exponents and the critical temperature.

Variational approximations to real space renormalisation group transformations were first introduced by Kadanoff (1975) as a method of selecting a 'best' transformation from a class of transformations obtained within a specified approximation scheme. Kadanoff (1975) and Kadanoff *et al* (1976) obtained remarkably accurate estimates for the critical exponents of Ising systems on d-dimensional hypercubic lattices ($d = 2$, 3, 4) from a simple lower bound approximation to the exact free energy. More recently, the Kadanoff lower bound renormalisation transformation (LBRT) has also been applied to more complex systems (Burkhardt 1976b, Burkhardt *et al* 1976, Dasgupta 1977, Burkhardt and Eisenriegler 1977, den Nijs and Knops 1977). In almost all cases, the approximation yielded results which compared very favourably with those obtained by more conventional methods. The reasons for this success are, however, not very clear.

The criterion proposed by Kadanoff (1975) to determine the 'best' fixed point out of the set allowed by the approximate recursion relations has been questioned by Knops (1977) (see also Barber 1977). In the case of the Ising model on a square lattice, both Burkhardt (1976a) and Knops (1977) showed that the fixed point found by Kadanoff (1975) possesses an additional relevant eigenvalue and is approached only within a restricted subspace of coupling constants. This subspace does not include the conventional near-neighbour Ising model on a square lattice unless a decimation transformation (Kadanoff and Houghton 1975) is first performed. There is an additional fixed point which does have the correct stability characteristics and can be reached from a starting Hamiltonian with only nearest-neighbour interactions, but the critical exponents associated with the new fixed point are not as accurate as those at the original point found by Kadanoff.

There is also evidence from Plischke and Austen (1976) which suggests that the Kadanoff approximation is not nearly so successful when applied to other two-dimensional lattices such as a triangular lattice. However the transformation used by these authors in this case could also be interpreted as describing a hypercubic lattice in $d = 1{\cdot}585$ dimensions since the parameter $z = 2^d$ is equal to 3 (see for example Katz *et al* 1977). The results that Plischke and Austen obtain are not unreasonable for this lower dimensionality. In view of this alternate interpretation we shall now describe a LBRT which can be used to investigate both the square and the triangular lattices in two dimensions.

The Hamiltonian of the system is taken to have the following form:

$$-\beta H = \sum_{\text{squares}} (K(S_1S_2+S_2S_3+S_3S_4+S_4S_1)+2L_1S_1S_3+2L_2S_2S_4+K_4S_1S_2S_3S_4) \qquad (1)$$

where the sum is over all elementary squares of a square lattice and the S are arranged as shown in figure 1. With the form of the Hamiltonian in equation (1) we may consider the following cases: (a) the choice $K>0$ and $L_1 = L_2 = K_4 = 0$ describes the ferromagnetic Ising model on a square lattice with nearest-neighbour interaction $2K$; (b) the choice $K = L_1 > 0$ and $L_2 = K_4 = 0$ describes the same model on a lattice which is topologically equivalent to a triangular lattice as shown in figure 2(a); and (c) finally the choice $K = L_2 > 0$ and $L_1 = K_4 = 0$ for even rows alternating with the choice in case (b) for odd rows also describes a triangular lattice as shown in figure 2(b). Hence different initial Hamiltonians permit us to describe both the square and triangular lattices within the same LBRT. In all cases the parameter $z = 2^d$ is equal to four corresponding to a dimensionality $d = 2$.

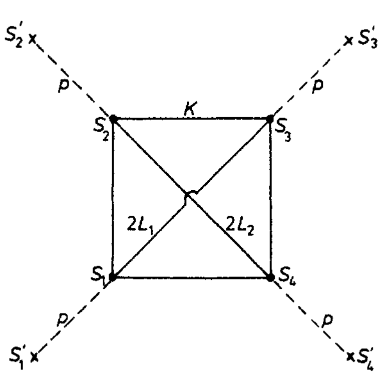

Figure 1. Basic cluster used for LBRT.

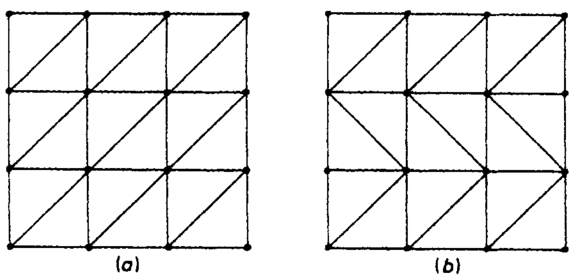

Figure 2. (a) Initial cluster for triangular lattice described by case (b); (b) initial cluster for triangular lattice described by case (c).

Kadanoff's original LBRT for the square lattice was restricted to the subspace $K = 2L_1 = 2L_2$. Burkhardt (1976*a*) and Knops (1977) considered the larger subspace $K \neq 2L_1 = 2L_2$ and found that the Kadanoff fixed point became unstable. For the triangular lattice described in case (*b*) above we have considered an even larger subspace in which $K \neq 2L_1 \neq 2L_2$ and we find that the Burkhardt fixed point has an additional relevant eigenvalue corresponding to perturbations in $L_1 - L_2$ indicating that it does not describe the critical behaviour of the triangular lattice. We have found an additional fixed point which is stable. The coordinates of all three fixed points are given in table 1 along with the eigenvalues of the linearised recursion relations at the fixed point. The leading thermal eigenvalue at this new critical fixed point is $\lambda_1^T = 1{\cdot}8976$ which yields $\nu = 1{\cdot}082$ and $\alpha = -0{\cdot}164$. The critical surface in the space of the parameters (K, L_1, L_2, K_4) intersects the line $K = L_1$, $L_2 = K_4 = 0$ at $K_c = 0{\cdot}167$ as compared with the exact value of $K_c = 0{\cdot}137$. The leading magnetic eigenvalue is $\lambda_1^H = 3{\cdot}7642$ which yields $\delta = 21{\cdot}819$. These results for the triangular lattice are not as good as those for the square lattice but are an improvement on those reported by Plischke and Austen.

Table 1. Critical fixed points of LBRT. λ^e and λ^o are the eigenvalues corresponding to operators of even and odd symmetry respectively.

Fixed point	K^*	$2L_1^*$	$2L_2^*$	K_4^*	p^*	λ^e	λ^o
Kadanoff	0·1397	0·1397	0·1397	−0·0069	0·766	2·0012	3·6688
						1·1151	1·7817
						1·1146	0·9582
						0·5056	0·7403
Burkhardt	0·1587	0·1077	0·1077	−0·0075	0·761	2·0245	3·6750
						1·1010	1·4591
						0·8871	0·9222
						0·4480	0·5415
Present calculation	0·1622	0·2744	0·0500	−0·0347	0·754	1·8976	3·7642
(case (*b*))						0·7625	1·7291
						0·6437	0·6737
						0·2683	0·3542

In case (*c*) above, the first iteration of the LBRT maps directly into the subspace $K = 2L_1 = 2L_2$ studied by Kadanoff (1975) and the critical properties are described by the fixed point found by Kadanoff. The critical exponents are $\alpha = 0{\cdot}002$ and $\delta = 15{\cdot}04$ with the critical value of K given by $K_c = 0{\cdot}139$ in excellent agreement with the exact values.

In summary, we have presented a LBRT for the Ising model in two dimensions which can be used to describe both the square and triangular lattices. The cluster considered in case (*c*) above gives excellent results for both the critical exponents and critical temperature of the triangular lattice and demonstrates the universal critical behaviour exhibited by these two types of lattice.

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